Chapter 2
Failure Models
Part 1: Introduction

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Reliability measures

In this chapter we introduce the following measures:

- The reliability (survivor) function $R(t)$
- The failure rate function $z(t)$
- The mean time to failure (MTTF)
- The mean residual life (MRL)

of a single item that is not repaired when it fails.

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Life distributions

The following life distributions are discussed:

- The exponential distribution
- The gamma distribution
- The Weibull distribution
- The normal distribution
- The lognormal distribution
- The Birnbaum-Saunders distribution
- The inverse Gaussian distributions

In addition we cover three discrete distributions:

- The binomial distribution
- The Poisson distribution
- The geometric distribution

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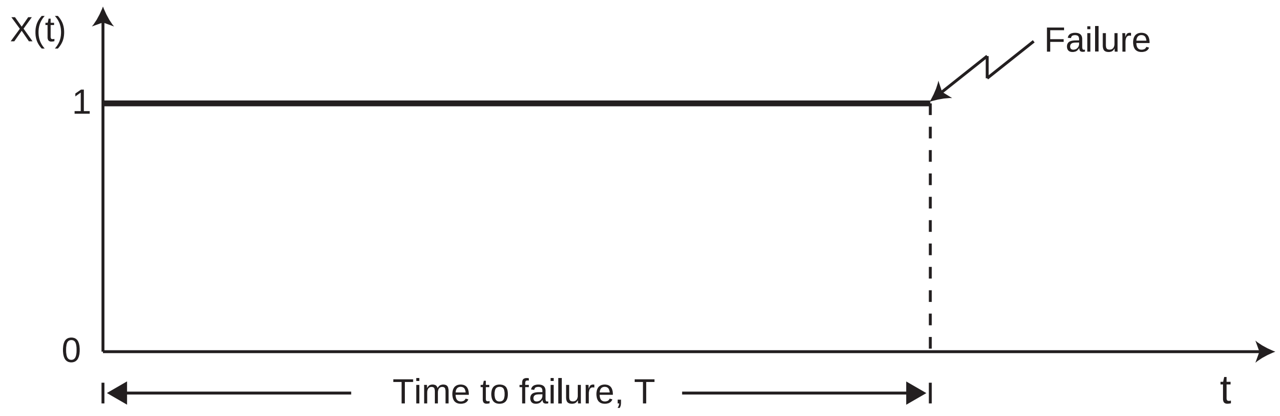
State variable



$$X(t) = \begin{cases} 1 & \text{if the item is functioning at time } t \\ 0 & \text{if the item is in a failed state at time } t \end{cases}$$

The state variable $X(t)$ and the time to failure T will generally be random variables.

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Time to failure

Different time concepts may be used, like

- Calendar time
- Operational time
- Number of kilometers driven by a car
- Number of cycles for a periodically working item
- Number of times a switch is operated
- Number of rotations of a bearing

In most applications we will assume that the time to failure T is a *continuous* random variable (Discrete variables may be approximated by a continuous variable)

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The *distribution function* of T is

$$F(t) = \Pr(T \leq t) = \int_0^t f(u)\, du \quad \text{for } t > 0$$

Note that

$F(t) =$ Probability that the item will fail within the interval $(0, t]$

Probability density function

The *probability density function* (pdf) of T is

$$f(t) = \frac{d}{dt} F(t) = \lim_{\Delta t \to \infty} \frac{F(t + \Delta t) - F(t)}{\Delta t} = \lim_{\Delta t \to \infty} \frac{\Pr(t < T \leq t + \Delta}{\Delta t}$$

When Δt is small, then

$$\boxed{\Pr(t < T \leq t + \Delta t) \approx f(t) \cdot \Delta t}$$

When we are standing at time $t = 0$ and ask: What is the probability that the item will fail in the interval $(t, t + \Delta t]$? The answer is approximately $f(t) \cdot \Delta t$

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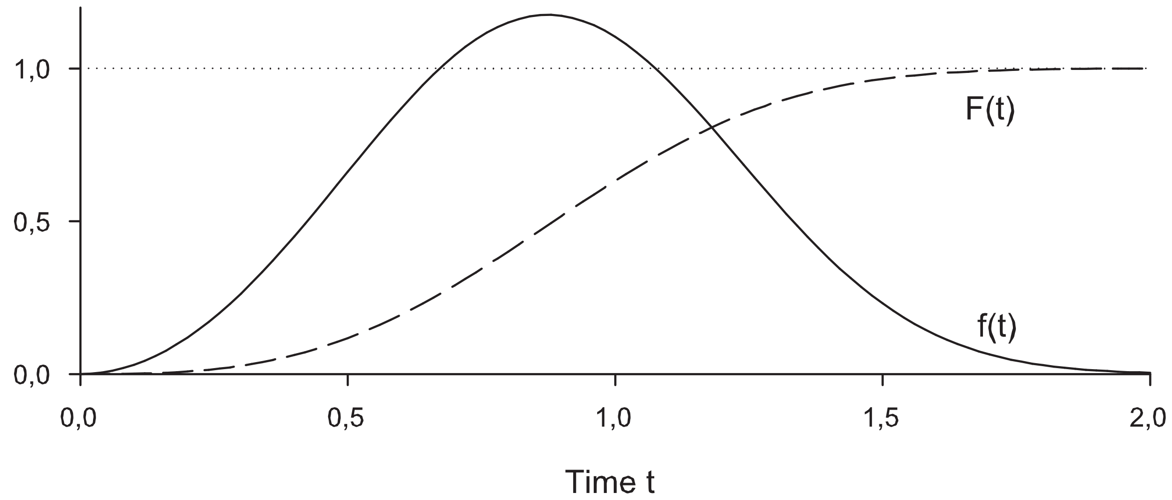
Distribution of T



- The area under the pdf-curve $(f(t))$ is always 1, $\int_0^\infty f(t)\,dt = 1$
- The area under the pdf-curve to the left of t is equal to $F(t)$
- The area under the pdf-curve between t_1 and t_2 is $F(t_2) - F(t_1) = \Pr(t_1 < T \leq t_2)$

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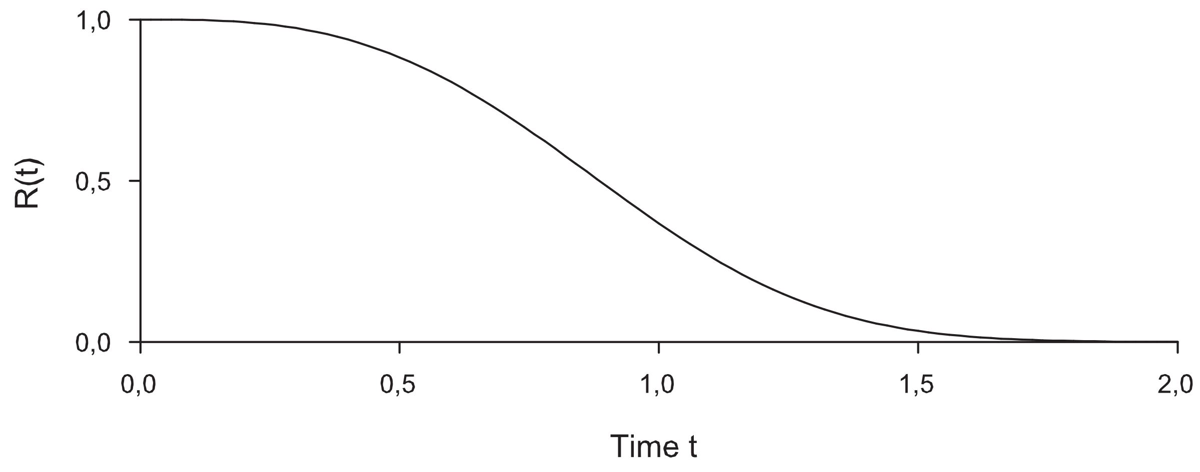

$$R(t) = \Pr(T > t) = 1 - F(t) = \int_t^{\infty} f(u)\,du$$

- $R(t) =$ The probability that the item will not fail in $(0, t]$
- $R(t) =$ The probability that the item will survive at least to time t
- $R(t)$ is also called the *survivor function* of the item

Failure rate function

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Consider the conditional probability

$$\Pr(t < T \leq t + \Delta t \mid T > t) = \frac{\Pr(t < T \leq t + \Delta t)}{\Pr(T > t)} = \frac{F(t + \Delta t) - F(t)}{R(t)}$$

The failure rate function of the item is

$$z(t) = \lim_{\Delta t \to 0} \frac{\Pr(t < T \leq t + \Delta t \mid T > t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \cdot \frac{1}{R(t)} = \frac{f(t)}{R(t)}$$

When Δt is small, we have

$$\boxed{\Pr(t < T \leq t + \Delta t \mid T > t) \approx z(t) \cdot \Delta t}$$

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Failure Rate Function (2)

- Note the difference between the failure rate function $z(t)$ and the probability density function $f(t)$.
- When we follow an item from time 0 and note that it is still functioning at time t, the probability that the item will fail during a short interval of length Δt after time t is $z(t) \cdot \Delta t$
- The failure rate function is a "property" of the item and is sometimes called the *force of mortality* (FOM) of the item.

Bathtub curve

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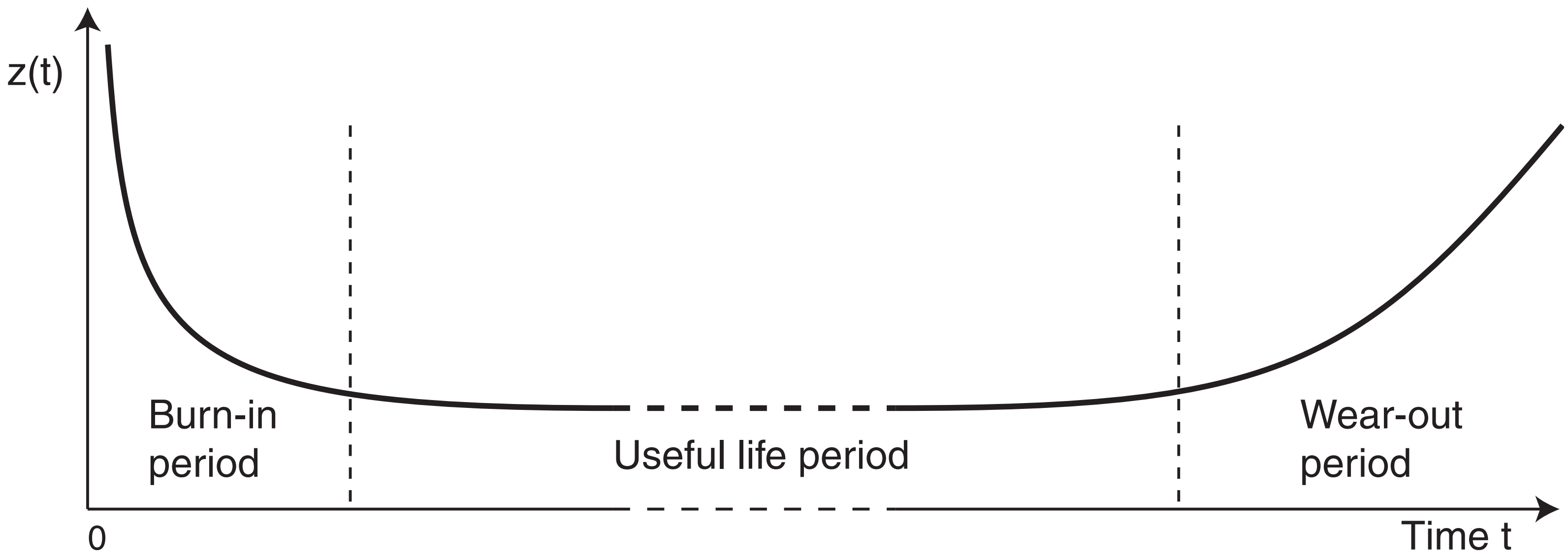

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Expressed by	$F(t)$	$f(t)$	$R(t)$	$z(t)$
$F(t) =$	–	$\int_0^t f(u)\,du$	$1 - R(t)$	$1 - \exp\left(-\int_0^t z(u)\,du\right)$
$f(t) =$	$\frac{d}{dt}F(t)$	–	$-\frac{d}{dt}R(t)$	$z(t) \cdot \exp\left(-\int_0^t z(u)\,du\right)$
$R(t) =$	$1 - F(t)$	$\int_t^\infty f(u)\,du$	–	$\exp\left(-\int_0^t z(u)\,du\right)$
$z(t) =$	$\frac{dF(t)/dt}{1 - F(t)}$	$\frac{f(t)}{\int_t^\infty f(u)\,du}$	$-\frac{d}{dt}\ln R(t)$	–

Mean time to failure

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The mean time to failure, MTTF, of an item is

$$\text{MTTF} = E(T) = \int_0^\infty t f(t)\, dt \tag{1}$$

Since $f(t) = -R'(t)$,

$$\text{MTTF} = -\int_0^\infty t R'(t)\, dt$$

By partial integration

$$\text{MTTF} = -\left[tR(t)\right]_0^\infty + \int_0^\infty R(t)\, dt$$

If MTTF $< \infty$, it can be shown that $\left[tR(t)\right]_0^\infty = 0$. In that case

$$\boxed{\text{MTTF} = \int_0^\infty R(t)\, dt} \tag{2}$$

It is often easier to determine MTTF by (2) than by (1).

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Consider an item with survivor function

$$R(t) = \frac{1}{(0.2\,t + 1)^2} \quad \text{for } t \geq 0$$

where the time t is measured in months. The probability density function is

$$f(t) = -R'(t) = \frac{0.4}{(0.2\,t + 1)^3}$$

and the failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \frac{0.4}{0.2\,t + 1}$$

The mean time to failure is:

$$\text{MTTF} = \int_0^\infty R(t)\,dt = 5 \text{ months}$$

Median

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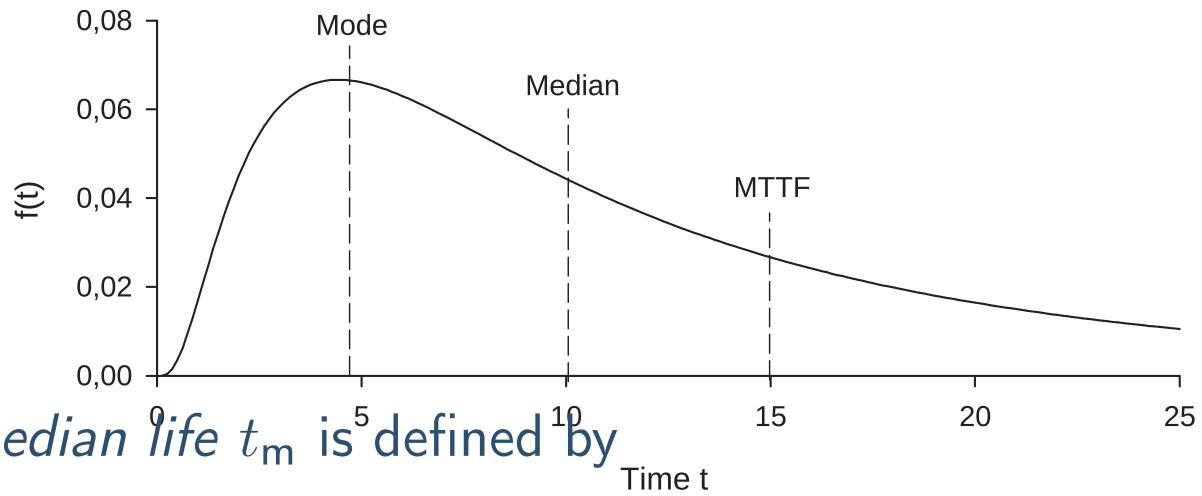

The *median life* t_m is defined by

$$R(t_m) = 0.50$$

The median divides the distribution in two halves. The item will fail before time t_m with 50% probability, and will fail after time t_m with 50% probability.

The *mode* of a life distribution is the most likely failure time, that is, the time t_{mode} where the probability density function $f(t)$ attains its maximum.:

Mode

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The *mode* of a life distribution is the most likely failure time, that is, the time t_{mode} where the probability density function $f(t)$ attains its maximum.:

Mean residual life

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Consider an item that is put into operation at time $t = 0$ and is still functioning at time t. The probability that the item of age t survives an additional interval of length x is

$$R(x \mid t) = \Pr(T > x + t \mid T > t) = \frac{\Pr(T > x + t)}{\Pr(T > t)} = \frac{R(x + t)}{R(t)}$$

$R(x \mid t)$ is called the *conditional survivor function* of the item at age t.

The *mean residual (or, remaining) life*, $\text{MRL}(t)$, of the item at age t is

$$\text{MRL}(t) = \mu(t) = \int_0^\infty R(x \mid t)\, dx = \frac{1}{R(t)} \int_t^\infty R(x)\, dx$$

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Example 2.2

Consider an item with failure rate function $z(t) = t/(t+1)$. The failure rate function is increasing and approaches 1 when $t \to \infty$. The corresponding survivor function is

$$\begin{aligned} R(t) &= \exp\left(-\int_0^t \frac{u}{u+1}\,du\right) = (t+1)\,e^{-t} \\ \text{MTTF} &= \int_0^\infty (t+1)\,e^{-t}\,dt = 2 \end{aligned}$$

The conditional survival function is

$$R(x \mid t) = \Pr(T > x + t \mid T > t) = \frac{(t+x+1)\,e^{-(t+x)}}{(t+1)\,e^{-t}} = \frac{t+x+1}{t+1}$$

The mean residual life is

$$\text{MRL}(t) = \int_0^\infty R(x \mid t)\,dx = 1 + \frac{1}{t+1}$$

We see that $\text{MRL}(t)$ is equal to 2 (= MTTF) when $t = 0$, that $\text{MRL}(t)$ is a decreasing function in t, and that $\text{MRL}(t) \to 1$ when $t \to \infty$.

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Binomial distribution

The *binomial situation* is defined by:

1. We have n independent trials.
2. Each trial has two possible outcomes A and A^*.
3. The probability $\Pr(A) = p$ is the same in all the n trials.

The trials in this situation are sometimes called *Bernoulli trials*. Let X denote the number of the n trials that have outcome A. The distribution of X is

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad \text{for } x = 0, 1, \ldots, n$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the binomial coefficient.

The distribution is called the *binomial distribution* (n, p), and we sometimes write $X \sim \text{bin}(n, p)$. The mean value and the variance of X are

$$E(X) = np \qquad \text{var}(X) = np(1-p)$$

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Geometric distribution

Assume that we carry out a sequence of Bernoulli trials, and want to find the number Z of trials until the first trial with outcome A. If $Z = z$, this means that the first $(z-1)$ trials have outcome A^*, and that the first A will occur in trial z. The distribution of Z is

$$\Pr(Z = z) = (1-p)^{z-1} p \quad \text{for } z = 1, 2, \ldots$$

This distribution is called the *geometric distribution*. We have that

$$\Pr(Z > z) = (1-p)^z$$

The mean value and the variance of Z are

$$E(Z) = \frac{1}{p}$$

$$\text{var}(X) = \frac{1-p}{p^2}$$

The homogeneous Poisson process (1)

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Consider occurrences of a specific event $\mathcal{A}$, and assume that

1. The event $\mathcal{A}$ may occur at any time in the interval, and the probability of $\mathcal{A}$ occurring in the interval $(t, t + \Delta t]$ is independent of t and may be written as $\lambda \cdot \Delta t + o(\Delta t)$, where λ is a positive constant.
2. The probability of more that one event $\mathcal{A}$ in the interval $(t, t + \Delta t]$ is $o(\Delta t)$.
3. Let $(t_{11}, t_{12}], (t_{21}, t_{22}], \ldots$ be any sequence of disjoint intervals in the time period in question. Then the events "$\mathcal{A}$ occurs in $(t_{j1}, t_{j2}]$," $j = 1, 2, \ldots$, are independent.

Without loss of generality we let $t = 0$ be the starting point of the process.

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The Homogeneous Poisson Process (2)

Let $N(t)$ denote the number of times the event $\mathcal{A}$ occurs during the interval $(0,t]$. The stochastic process $\{N(t), t \geq 0\}$ is called a Homogeneous Poisson Process (HPP) with rate λ.

The distribution of $N(t)$ is

$$\Pr(N(t) = n) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t} \quad \text{for } n = 0, 1, 2, \ldots$$

The mean and the variance of $N(t)$ are

$$\begin{aligned} E(N(t)) &= \sum_{n=0}^{\infty} n \cdot \Pr(N(t) = n) = \lambda t \\ \text{var}(N(t)) &= \lambda t \end{aligned}$$

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Exponential distribution

Consider an item that is put into operation at time $t = 0$. Assume that the time to failure T of the item has probability density function (pdf)

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t > 0,\ \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

This distribution is called the *exponential distribution* with parameter λ, and we sometimes write $T \sim \exp(\lambda)$.

The survivor function of the item is

$$R(t) = \Pr(T > t) = \int_t^\infty f(u)\,du = e^{-\lambda t} \quad \text{for } t > 0$$

The mean and the variance of T are

$$\begin{aligned} \text{MTTF} &= \int_0^\infty R(t)\,dt = \int_0^\infty e^{-\lambda t}\,dt = \frac{1}{\lambda} \\ \text{var}(T) &= 1/\lambda^2 \end{aligned}$$

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Exponential distribution (2)

The failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

The failure rate function is hence constant and independent of time.
Consider the conditional survivor function

$$\begin{aligned} R(x \mid t) &= \Pr(T > t + x \mid T > t) = \frac{\Pr(T > t + x)}{\Pr(T > t)} \\ &= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = \Pr(T > x) = R(x) \end{aligned}$$

A new item, and a used item (that is still functioning), will therefore have the same probability of surviving a time interval of length t.
A used item is therefore stochastically *as good as new*.

Weibull distribution

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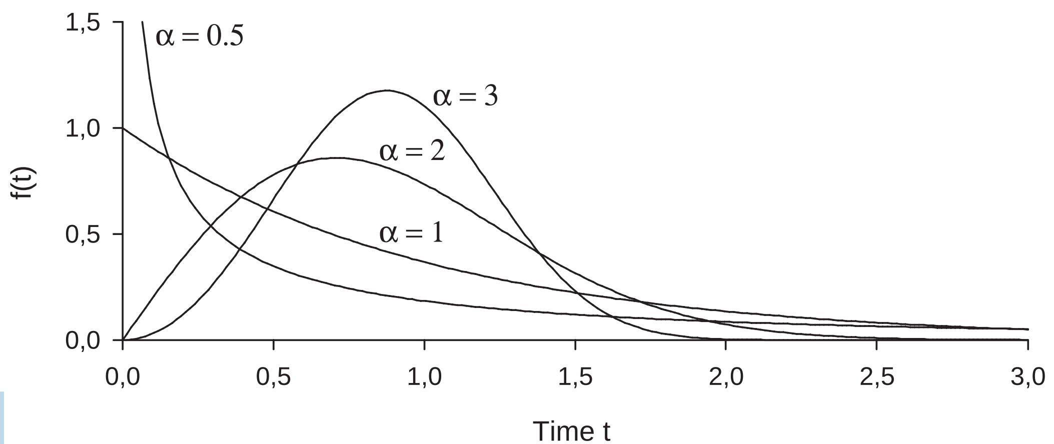
The time to failure T of an item is said to be Weibull distributed with parameters α and λ [$T \sim \text{Weibull}(\alpha, \lambda)$] if the distribution function is given by

$$F(t) = \Pr(T \leq t) = \begin{cases} 1 - e^{-(\lambda t)^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

The corresponding probability density function (pdf) is

$$f(t) = \frac{d}{dt}F(t) = \begin{cases} \alpha \lambda^\alpha t^{\alpha-1} e^{-(\lambda t)^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$



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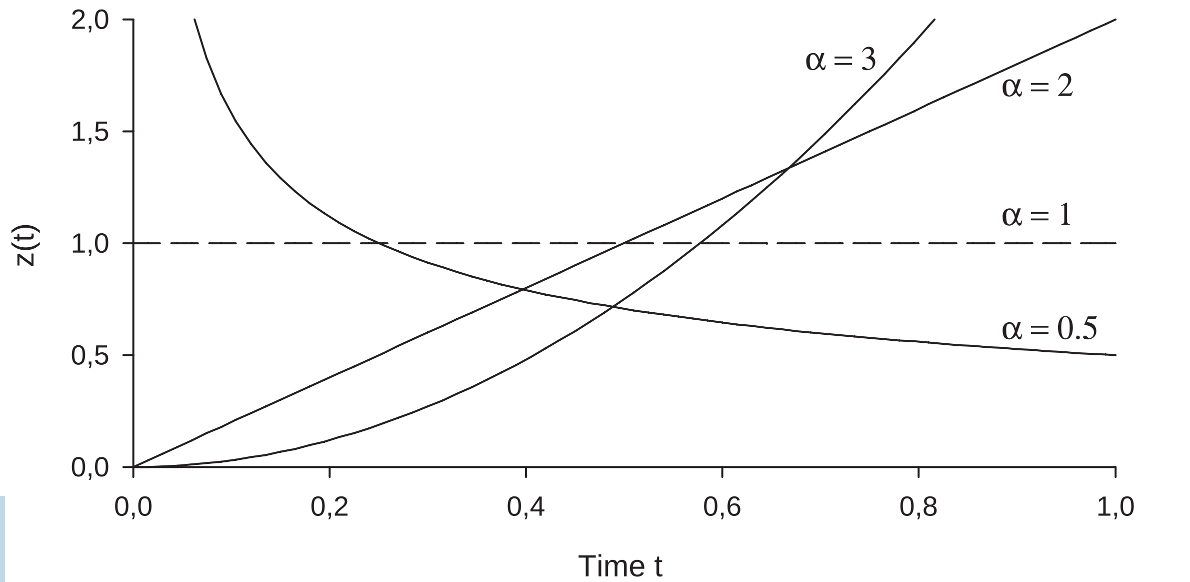
Weibull distribution (2)

The survivor function is

$$R(t) = \Pr(T > 0) = e^{-(\lambda t)^\alpha} \quad \text{for} \quad t > 0$$

and the failure rate function is

$$z(t) = \frac{f(t)}{R(t)} = \alpha \lambda^\alpha t^{\alpha - 1} \quad \text{for} \quad t > 0$$



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Weibull Distribution (3)

The mean time to failure is

$$\text{MTTF} = \int_0^\infty R(t)\,dt = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha}+1\right)$$

The median life t_{m} is

$$R(t_{\text{m}}) = 0.50 \quad \Rightarrow \quad t_{\text{m}} = \frac{1}{\lambda}\,(\ln 2)^{1/\alpha}$$

The variance of T is

$$\text{var}(T) = \frac{1}{\lambda^2}\left(\Gamma\left(\frac{2}{\alpha}+1\right) - \Gamma^2\left(\frac{1}{\alpha}+1\right)\right)$$

Note that $\text{MTTF}/\sqrt{\text{var}(T)}$ is independent of λ.