Chapter 2
Failure Models
Part 1: Introduction

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Introduction

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Time to failure
Distr. function
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Life distributions

Marvin Rausand, March 14, 2006

System Reliability Theory (2nd ed), Wiley, 2004 – 2 / 31
In this chapter we introduce the following measures:

- The reliability (survivor) function $R(t)$
- The failure rate function $\lambda(t)$
- The mean time to failure (MTTF)
- The mean residual life (MRL)

of a single item that is not repaired when it fails.
The following life distributions are discussed:

- The exponential distribution
- The gamma distribution
- The Weibull distribution
- The normal distribution
- The lognormal distribution
- The Birnbaum-Saunders distribution
- The inverse Gaussian distributions

In addition we cover three discrete distributions:

- The binomial distribution
- The Poisson distribution
- The geometric distribution
The state variable $X(t)$ and the time to failure $T$ will generally be random variables.
Different time concepts may be used, like

- Calendar time
- Operational time
- Number of kilometers driven by a car
- Number of cycles for a periodically working item
- Number of times a switch is operated
- Number of rotations of a bearing

In most applications we will assume that the time to failure $T$ is a \textit{continuous} random variable (Discrete variables may be approximated by a continuous variable)
The distribution function of $T$ is

$$F(t) = \Pr(T \leq t) = \int_0^t f(u) \, du \quad \text{for } t > 0$$

Note that

$F(t) =$ Probability that the item will fail within the interval $(0, t]$
The probability density function (pdf) of $T$ is

$$f(t) = \frac{d}{dt} F(t) = \lim_{\Delta t \to \infty} \frac{F(t + \Delta t) - F(t)}{\Delta t} = \lim_{\Delta t \to \infty} \frac{\Pr(t < T \leq t + \Delta t)}{\Delta t}$$

When $\Delta t$ is small, then

$$\Pr(t < T \leq t + \Delta t) \approx f(t) \cdot \Delta t$$

When we are standing at time $t = 0$ and ask: What is the probability that the item will fail in the interval $(t, t + \Delta t]$? The answer is approximately $f(t) \cdot \Delta t$
The area under the pdf-curve \((f(t))\) is always 1,
\[
\int_{0}^{\infty} f(t) \, dt = 1
\]
- The area under the pdf-curve to the left of \(t\) is equal to \(F(t)\)
- The area under the pdf-curve between \(t_1\) and \(t_2\) is
\[
F(t_2) - F(t_1) = \Pr(t_1 < T \leq t_2)
\]
**Reliability Function**

- **Introduction**
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  - Life distributions
  - State variable
  - Time to failure
  - Distr. function
  - Probability density
  - Distribution of $T$

- **Reliability funct.**
  - Failure rate
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- **Discrete distributions**

- **Life distributions**

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The probability that the item will not fail in $(0, t]$

$R(t) = \Pr(T > t) = 1 - F(t) = \int_t^{\infty} f(u) \, du$

- $R(t)$ = The probability that the item will not fail in $(0, t]$
- $R(t)$ = The probability that the item will survive at least to time $t$
- $R(t)$ is also called the *survivor function* of the item
Consider the conditional probability

\[
\Pr(t < T \leq t + \Delta t \mid T > t) = \frac{\Pr(t < T \leq t + \Delta t)}{\Pr(T > t)} = \frac{F(t + \Delta t) - F(t)}{R(t)}
\]

The failure rate function of the item is

\[
z(t) = \lim_{\Delta t \to 0} \frac{\Pr(t < T \leq t + \Delta t \mid T > t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \cdot \frac{1}{R(t)} = \frac{f(t)}{R(t)}
\]

When \(\Delta t\) is small, we have

\[
\Pr(t < T \leq t + \Delta t \mid T > t) \approx z(t) \cdot \Delta t
\]
Note the difference between the failure rate function $z(t)$ and the probability density function $f(t)$.

When we follow an item from time 0 and note that it is still functioning at time $t$, the probability that the item will fail during a short interval of length $\Delta t$ after time $t$ is $z(t) \cdot \Delta t$.

The failure rate function is a “property” of the item and is sometimes called the force of mortality (FOM) of the item.
The Bathtub curve is a graphical representation of the failure rate of a system over time. It is divided into three distinct periods:

1. **Burn-in period**: During this period, the failure rate is high due to defects that are not immediately apparent. This is often the time during which hardware is subjected to initial testing and debugging.
2. **Useful life period**: This is the phase where the failure rate is relatively constant and low. The system is operating at its expected level of reliability.
3. **Wear-out period**: As time progresses, the failure rate increases due to wear and tear. This is the period where components begin to fail at an accelerating rate.

The graph shows the relationship between time and failure rate, with the x-axis representing time (t) and the y-axis representing the failure rate (z(t)). The curve typically starts high, decreases to a minimum, and then increases again as the system enters the wear-out phase.
### Some formulas

<table>
<thead>
<tr>
<th>Expressed by</th>
<th>( F(t) )</th>
<th>( f(t) )</th>
<th>( R(t) )</th>
<th>( z(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(t) = )</td>
<td>(-\int_{0}^{t} f(u) , du)</td>
<td>(1 - R(t))</td>
<td>(1 - \exp \left(-\int_{0}^{t} z(u) , du\right))</td>
<td></td>
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<tr>
<td>( f(t) = )</td>
<td>(\frac{d}{dt} F(t))</td>
<td>(-\frac{d}{dt} R(t))</td>
<td>(z(t) \cdot \exp \left(-\int_{0}^{t} z(u) , du\right))</td>
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<tr>
<td>( R(t) = )</td>
<td>(1 - F(t))</td>
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<td>(-\exp \left(-\int_{0}^{t} z(u) , du\right))</td>
<td></td>
</tr>
<tr>
<td>( z(t) = )</td>
<td>(\frac{dF(t)/dt}{1 - F(t)})</td>
<td>(\frac{f(t)}{\int_{t}^{\infty} f(u) , du})</td>
<td>(-\frac{d}{dt} \ln R(t))</td>
<td></td>
</tr>
</tbody>
</table>
Mean time to failure

The mean time to failure, MTTF, of an item is

\[ \text{MTTF} = E(T) = \int_0^\infty t f(t) \, dt \]  

(1)

Since \( f(t) = -R'(t) \),

\[ \text{MTTF} = - \int_0^\infty t R'(t) \, dt \]

By partial integration

\[ \text{MTTF} = - [tR(t)]_0^\infty + \int_0^\infty R(t) \, dt \]

If \( \text{MTTF} < \infty \), it can be shown that \([tR(t)]_0^\infty = 0\). In that case

\[ \text{MTTF} = \int_0^\infty R(t) \, dt \]  

(2)

It is often easier to determine MTTF by (2) than by (1).
Example 2.1

Consider an item with survivor function

\[ R(t) = \frac{1}{(0.2t + 1)^2} \quad \text{for } t \geq 0 \]

where the time \( t \) is measured in months. The probability density function is

\[ f(t) = -R'(t) = \frac{0.4}{(0.2t + 1)^3} \]

and the failure rate function is

\[ z(t) = \frac{f(t)}{R(t)} = \frac{0.4}{0.2t + 1} \]

The mean time to failure is:

\[ \text{MTTF} = \int_{0}^{\infty} R(t) \, dt = 5 \text{ months} \]
The **median life** $t_m$ is defined by

$$R(t_m) = 0.50$$

The median divides the distribution in two halves. The item will fail before time $t_m$ with 50% probability, and will fail after time $t_m$ with 50% probability.

The **mode** of a life distribution is the most likely failure time, that is, the time $t_{mode}$ where the probability density function $f(t)$ attains its maximum.
The *mode* of a life distribution is the most likely failure time, that is, the time $t_{\text{mode}}$ where the probability density function $f(t)$ attains its maximum.
Consider an item that is put into operation at time $t = 0$ and is still functioning at time $t$. The probability that the item of age $t$ survives an additional interval of length $x$ is

$$R(x \mid t) = \Pr(T > x + t \mid T > t) = \frac{\Pr(T > x + t)}{\Pr(T > t)} = \frac{R(x + t)}{R(t)}$$

$R(x \mid t)$ is called the *conditional survivor function* of the item at age $t$.

The *mean residual (or, remaining) life*, $\text{MRL}(t)$, of the item at age $t$ is

$$\text{MRL}(t) = \mu(t) = \int_0^\infty R(x \mid t) \, dx = \frac{1}{R(t)} \int_t^\infty R(x) \, dx$$
Example 2.2
Consider an item with failure rate function \( z(t) = \frac{t}{t+1} \). The failure rate function is increasing and approaches 1 when \( t \to \infty \). The corresponding survivor function is

\[
R(t) = \exp \left( - \int_0^t \frac{u}{u+1} \, du \right) = (t+1) e^{-t}
\]

\[
\text{MTTF} = \int_0^\infty (t+1) e^{-t} \, dt = 2
\]

The conditional survival function is

\[
R(x \mid t) = \Pr(T > x + t \mid T > t) = \frac{(t+x+1) e^{-(t+x)}}{(t+1) e^{-t}} = \frac{t+x+1}{t+1}
\]

The mean residual life is

\[
\text{MRL}(t) = \int_0^\infty R(x \mid t) \, dx = 1 + \frac{1}{t+1}
\]

We see that \( \text{MRL}(t) \) is equal to 2 (= MTTF) when \( t = 0 \), that \( \text{MRL}(t) \) is a decreasing function in \( t \), and that \( \text{MRL}(t) \to 1 \) when \( t \to \infty \).
Discrete distributions
Binomial distribution

The *binomial situation* is defined by:

1. We have $n$ independent trials.
2. Each trial has two possible outcomes $A$ and $A^*$.
3. The probability $\Pr(A) = p$ is the same in all the $n$ trials.

The trials in this situation are sometimes called *Bernoulli trials*. Let $X$ denote the number of the $n$ trials that have outcome $A$. The distribution of $X$ is

$$
\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \quad \text{for } x = 0, 1, \ldots, n
$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the binomial coefficient.

The distribution is called the *binomial distribution* $(n, p)$, and we sometimes write $X \sim \text{bin}(n, p)$. The mean value and the variance of $X$ are

$$
E(X) = np \quad \text{var}(X) = np(1 - p)
$$
**Geometric distribution**

Assume that we carry out a sequence of Bernoulli trials, and want to find the number $Z$ of trials until the first trial with outcome $A$. If $Z = z$, this means that the first $(z - 1)$ trials have outcome $A^*$, and that the first $A$ will occur in trial $z$. The distribution of $Z$ is

$$
\Pr(Z = z) = (1 - p)^{z-1}p \quad \text{for } z = 1, 2, \ldots
$$

This distribution is called the *geometric distribution*. We have that

$$
\Pr(Z > z) = (1 - p)^z
$$

The mean value and the variance of $Z$ are

$$
E(Z) = \frac{1}{p}
$$

$$
\text{var}(X) = \frac{1 - p}{p^2}
$$
Consider occurrences of a specific event $A$, and assume that

1. The event $A$ may occur at any time in the interval, and the probability of $A$ occurring in the interval $(t, t + \Delta t]$ is independent of $t$ and may be written as $\lambda \cdot \Delta t + o(\Delta t)$, where $\lambda$ is a positive constant.

2. The probability of more than one event $A$ in the interval $(t, t + \Delta t]$ is $o(\Delta t)$.

3. Let $(t_{11}, t_{12}), (t_{21}, t_{22}), \ldots$ be any sequence of disjoint intervals in the time period in question. Then the events “$A$ occurs in $(t_{j1}, t_{j2}]$, $j = 1, 2, \ldots$,” are independent.

Without loss of generality we let $t = 0$ be the starting point of the process.
Let \( N(t) \) denote the number of times the event \( A \) occurs during the interval \((0, t]\). The stochastic process \( \{N(t), t \geq 0\} \) is called a Homogeneous Poisson Process (HPP) with rate \( \lambda \).

The distribution of \( N(t) \) is

\[
Pr(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \text{for } n = 0, 1, 2, \ldots
\]

The mean and the variance of \( N(t) \) are

\[
E(N(t)) = \sum_{n=0}^{\infty} n \cdot Pr(N(t) = n) = \lambda t
\]

\[
\text{var}(N(t)) = \lambda t
\]
Life distributions
Exponential distribution

Consider an item that is put into operation at time $t = 0$. Assume that the time to failure $T$ of the item has probability density function (pdf)

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t > 0, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

This distribution is called the *exponential distribution* with parameter $\lambda$, and we sometimes write $T \sim \text{exp}(\lambda)$.

The survivor function of the item is

$$R(t) = \Pr(T > t) = \int_t^{\infty} f(u) \, du = e^{-\lambda t} \quad \text{for } t > 0$$

The mean and the variance of $T$ are

$$\text{MTTF} = \int_0^{\infty} R(t) \, dt = \int_0^{\infty} e^{-\lambda t} \, dt = \frac{1}{\lambda}$$

$$\text{var}(T) = \frac{1}{\lambda^2}$$
Exponential distribution (2)

The failure rate function is

\[ z(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \]

The failure rate function is hence constant and independent of time.

Consider the conditional survivor function

\[ R(x \mid t) = \Pr(T > t + x \mid T > t) = \frac{\Pr(T > t + x)}{\Pr(T > t)} \]

\[ = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} = \Pr(T > x) = R(x) \]

A new item, and a used item (that is still functioning), will therefore have the same probability of surviving a time interval of length \( t \).

A used item is therefore stochastically as good as new.
Weibull distribution

The time to failure $T$ of an item is said to be Weibull distributed with parameters $\alpha$ and $\lambda$ [$T \sim \text{Weibull}(\alpha, \lambda)$] if the distribution function is given by

$$F(t) = \Pr(T \leq t) = \begin{cases} 1 - e^{-(\lambda t)^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

The corresponding probability density function (pdf) is

$$f(t) = \frac{d}{dt} F(t) = \begin{cases} \alpha \lambda \alpha t^{\alpha-1} e^{-(\lambda t)^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$
The survivor function is

\[ R(t) = \Pr(T > 0) = e^{-(\lambda t)^\alpha} \quad \text{for } t > 0 \]

and the failure rate function is

\[ z(t) = \frac{f(t)}{R(t)} = \alpha \lambda^\alpha t^{\alpha - 1} \quad \text{for } t > 0 \]
The mean time to failure is

\[ \text{MTTF} = \int_0^\infty R(t) \, dt = \frac{1}{\lambda} \Gamma \left( \frac{1}{\alpha} + 1 \right) \]

The median life \( t_m \) is

\[ R(t_m) = 0.50 \quad \Rightarrow \quad t_m = \frac{1}{\lambda} (\ln 2)^{1/\alpha} \]

The variance of \( T \) is

\[ \text{var}(T) = \frac{1}{\lambda^2} \left( \Gamma \left( \frac{2}{\alpha} + 1 \right) - \Gamma^2 \left( \frac{1}{\alpha} + 1 \right) \right) \]

Note that \( \text{MTTF}/\sqrt{\text{var}(T)} \) is independent of \( \lambda \).