Chapter 9
Reliability of Maintained Systems

Part 2: Optimization of Replacement Intervals

Marvin Rausand

Department of Production and Quality Engineering
Norwegian University of Science and Technology
marvin.rausand@ntnu.no
Two Different Criteria

Age Replacement

Block Replacement

Condition-Based Replacement

PF Intervals
Two Different Criteria

- **Time based replacement**
  - Age replacement
  - Block replacement

- **Condition based replacement**
  - Continuous deterioration
  - Deterioration following a shock

![Graph showing the two criteria with phases: Burn-in period, Useful life period, Wear-out period.](image-url)
Age Replacement
Age Replacement Policy

- Planned replacement, cost = c
- Failure, replacement cost = c+k
- Planned replacement, cost = c

\[ c \quad = \quad \text{Planned replacement cost} \]
\[ c + k \quad = \quad \text{Unplanned replacement cost} \]
\[ t_0 \quad = \quad \text{Planned replacement age} \]
Age Replacement - Examples

- Engine drive chain of (some) automobiles
  - On old Volvos the drive chain should be replaced every 80,000 km
- Oil and oil filters in automobiles
- Parts in airplanes
Age Replacement

- The mean time between replacements:

\[
MTBR(t_0) = \int_0^{t_0} tf(t) \, dt + t_0 \cdot Pr(T \geq t_0)
\]

\[
= \int_0^{t_0} (1 - F(t)) \, dt
\]

- The mean cost per replacement period:

\[
c + k \cdot Pr(\text{failure}) = c + k \cdot Pr(T < t_0)
\]

\[
= c + k \cdot F(t_0)
\]
Cost Optimization Criterion (1)

The mean cost per time unit, \( C_A(t_0) \), is determined from

\[
C_A(t_0) \cdot MTBR(t_0) = c + k \cdot F(t_0)
\]

The cost optimal replacement interval \( t_0 \) can therefore be found by minimizing

\[
C_A(t_0) = \frac{c + k \cdot F(t_0)}{\int_0^{t_0} (1 - F(t)) \, dt}
\]

with respect to \( t_0 \).
Cost Optimization Criterion (2)

If we let \( t_0 \to \infty \), we get

\[
C_A(\infty) = \lim_{t_0 \to \infty} C_A(t_0) = \frac{c + k}{\int_0^\infty (1 - F(t)) \, dt} = \frac{c + k}{MTTF}
\]

which is an obvious result, since no age replacements will take place. Now consider

\[
\frac{C_A(t_0)}{C_A(\infty)} = \frac{c + k \cdot F(t_0)}{\int_0^{t_0} (1 - F(t)) \, dt} \cdot \frac{MTTF}{c + k} = \frac{1 + r \cdot F(t_0)}{\int_0^{t_0} (1 - F(t)) \, dt} \cdot \frac{MTTF}{1 + r}
\]

where \( r = \frac{k}{c} \).
The ratio

\[ \frac{C_A(t_0)}{C_A(\infty)} \]

may be used as a measure of the cost efficiency of the age replacement policy with replacement interval \( t_0 \). A low value of the ratio \( C_A(t_0)/C_A(\infty) \) indicates a high cost efficiency.
Example 9.9 (1)

Let $T \sim \text{Weibull}(\alpha, \lambda)$ where $\alpha > 1$. We get

$$\frac{C_A(t_0)}{C_A(\infty)} = 1 + r \left( 1 - e^{-\left(\lambda t_0\right)^\alpha} \right) \cdot \frac{\Gamma(1/\alpha + 1)/\lambda}{1 + r} \int_0^{t_0} e^{-\left(\lambda t\right)^\alpha} \, dt$$

By introducing $x_0 = \lambda t_0$, we get

$$\frac{\tilde{C}_A(x_0)}{C_A(\infty)} = 1 + r \left( 1 - e^{-x_0^\alpha} \right) \cdot \frac{\Gamma(1/\alpha + 1)}{1 + r} \int_0^{x_0} e^{-x^\alpha} \, dx$$

We have to use a computer to find the $x_0$ that minimizes $\frac{\tilde{C}_A(x_0)}{C_A(\infty)}$.
Example 9.9 (2)

The ratio $\frac{\tilde{C}_A(x_0)}{C_A(\infty)}$ as a function of $x_0$ for the Weibull distribution with shape parameter $\alpha = 3$, and $r = 3, 5, \text{ and } 10$. 

The graph shows the ratio for $r = 3, 5, \text{ and } 10$, with $x_0 = \lambda \cdot t_0$. The area below the curves represents the preventive replacement area, and the area above represents the only corrective replacement area.
Availability Criterion (1)

In some applications we may be interested in finding the age replacement interval $t_0$ that minimizes the *average unavailability* of the item.

Mean downtime in a replacement period

$$MDT(t_0) = MDT_F \cdot F(t_0) + MDT_P \cdot (1 - F(t_0))$$

where:
- $MDT_F =$ the mean downtime due to a failure
- $MDT_P =$ the mean downtime due to a preventive replacement
The mean time between replacements is

\[
\text{MTBR}(t_0) = \int_0^{t_0} (1 - F(t)) \, dt + \text{MDT}_F \cdot F(t_0) + \text{MDT}_P \cdot (1 - F(t_0))
\]

The optimal \( t_0 \) then is the \( t_0 \) that minimizes

\[
\bar{A}_{av}(t_0) = \frac{\text{MDT}(t_0)}{\text{MTBR}(t_0)}
\]
Age Replacement Problems

- Has to assume that the item is replaced with an “as good as new” item (not worse and not better)
- Has to monitor the age of the item (may be difficult to administer when we have many items)
- Maintenance actions will be spread out in time
Block Replacement
Block Replacement

- The item is replaced at regular time intervals $t_0, 2t_0, \ldots$ regardless of age
- If an item fails within an interval, it is repaired (minimal, imperfect, or perfect)
- Let $N(t_0)$ be the number of failures/repairs within an interval of length $t_0$
- Let $W(t_0) = E(N(t_0))$
- The cost optimal replacement interval is found by minimizing

$$C_B(t_0) = \frac{c + k \cdot W(t_0)}{t_0}$$

with respect to $t_0$
Block Replacement - Example

- Replace all light bulbs in a building every 1. January
- Better examples??
Example 9.10 (1)

Assume that the replacement interval is so short that the probability of 2 or more replacements within $(0, t_0)$ is negligible. In this case

$$W(t_0) = E(N(t_0)) \approx \Pr(N(t_0) = 1) = F(t_0)$$

The average cost $C_B(t_0)$ per time unit is

$$C_B(t_0) \approx \frac{c + k \cdot F(t_0)}{t_0}$$

The minimum of $C_B(t_0)$ may be found by solving $dC_B(t_0)/dt_0 = 0$, which gives

$$\frac{c}{k} + F(t_0) = t \cdot F'(t_0)$$
Example 9.10 (2)

Let $t \sim \text{Weibull} \left( \alpha, \lambda \right)$ where $\alpha > 1$. In this case we get

$$\frac{c}{k} + 1 - e^{(\lambda t_0)^\alpha} = t_0 \cdot \alpha \lambda^{\alpha} t_0^{\alpha - 1} e^{-(\lambda t_0)^\alpha}$$

which can be written as

$$\frac{c}{k} + 1 = \left( 1 + \alpha (\lambda t_0)^\alpha \right) \cdot e^{-(\lambda t_0)^\alpha}$$

For this model to be realistic the preventive replacement cost $c$ must be small compared to the corrective replacement cost $k$. 
The optimal replacement interval $t_0$ as a function of the shape parameter $\alpha$ of the Weibull distribution. The optimal value $t_0$ is equal to $h \cdot \text{MTTF}$. 

$h$ vs $\alpha$ graph
Summary

- The block replacement policy is easier to administer than the age replacement policy
- We may have to replace rather new items
- Different types of “intermediate” repair may be considered (minimal, imperfect, perfect)
- Problems related to number of spares available may be considered
- The policy may be optimized with respect to cost and/or availability
Condition-Based Replacement
Condition-Based (1)

- $Y(t) =$ the deterioration of the item as a function of time $t$
- The deterioration is modelled as a stochastic process; usually as a gamma process, or a Wiener process
The item is inspected, and the deterioration “measured”, at times $t_1, t_2, \ldots$

- When a measurement is $\geq y_p$ the item is preventively replaced at cost $c$
- When a measurement is $\geq y_c$ the item is correctively replaced at cost $c + k$
Example 9.14 (1)

- Deterioration $Y(t)$ is modelled as a gamma process
- Inspection interval: $\tau$
- Deterioration in intervals $\Delta Y_1, \Delta Y_2, \ldots$ are independent and gamma distributed with distribution function $F(y)$, and mean $\mu \tau$
- Let $n_p =$ mean number of inspections until the deterioration reaches the threshold $y_p$, that is $n_p \cdot \mu \tau \approx y_p$
- The crossing will be detected in inspection $\tilde{n}_p$ where

$$\tilde{n}_p = \frac{y_p}{\mu \tau} + 1$$
Example 9.14 (2)

- The mean time between replacements is

\[
MTBR(\tau) = \left( \frac{y_p}{\mu \tau} + 1 \right) \cdot \tau
\]

- The average cost per replacement cycle is

\[
c + k_i \cdot n_p + k \cdot \Pr(\text{failure})
\]

\[
= c + k_i \cdot n_p + k \cdot \Pr(\Delta Y > (y_c - y_p))
\]

\[
= c + k_i \cdot n_p + k \cdot [1 - F(y_c - y_p)]
\]
Gamma Process (1)

In some applications it has been found to be realistic to model the deterioration as a \textit{gamma stochastic process} \{Y(t), t \geq 0\}, with the following characteristics:

1. \(Y(0) = 0\).

2. The process \{Y(t), t \geq 0\} has independent increments.

3. For all \(0 \leq s < t\), the random variable \(Y(t) - Y(s)\) has a gamma distribution with parameters \((\alpha(t-s), \beta)\), with probability density function

\[
f_{(s,t)}(y) = \frac{\beta}{\Gamma(\alpha(t-s))} (\beta y)^{\alpha(t-s)-1} e^{-\beta y} \quad \text{for } y \geq 0
\]
The mean deterioration in the interval \((s, t)\) is:

\[
E [Y(t) - Y(s)] = \frac{\alpha}{\beta} (t - s)
\]

When using the gamma process, the mean deterioration is therefore a linear function of time with deterioration speed (slope) \(\alpha/\beta\).
PF Intervals
PF Intervals (1)

- The item is inspected after regular intervals of length $\tau$
- Shock occur according to a homogeneous Poisson process with rate $\lambda$
- A potential failure (following a shock) is detectable at time $P$
- The item is functionally failed at time $F$
PF Intervals (2)

- $C_P =$ the cost of a preventive replacement
- $C_C =$ the cost of a corrective replacement
- $C_I =$ the cost of an inspection

Objective:

To find the test interval that minimizes the total cost!

The following situations are treated in the book:
- Deterministic PF interval and repair time, perfect inspection
- Stochastic PF interval, deterministic repair time, and non-perfect inspection
Example 9.16

- Cracks in (railway) rails
- Frequency of initiated cracks depends on traffic load, rail material, rail geometry, particles on the rails, shocks from trains with non-circular wheels, and so on
- Inspection by a special rail-car with ultrasonic inspection equipment (implies high cost)
- Detection probability depends on the depth of the crack
- PF interval is the time interval from the crack is observable until a critical failure occurs
- A critical failure may involve derailment and fatalities (i.e., \( C_C \) difficult to assess)